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The effect of centrifugal acceleration on axisymmetric convection in a shallow rotating cylinder or annulus

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This paper considers the nature of stationary axisymmetric convection in a shallow rotating annulus or cylinder heated from below. The horizontal and vertical walls of the container are assumed to be perfect conductors and insulators respectively. The critical regime is considered in which the circulations due to the centrifugal force and the vertical gravitational instability are of equal significance. Among the effects of the centrifugal acceleration on the gravitational instability are a shift in the value of the Rayleigh number at which instability sets in, a modification in the distribution and amplitude of convection cells in the container and, most significantly, a smooth transition to finite amplitude convection as the Rayleigh number is increased.

1. Introduction

The classical theory of Bénard convection in an infinite rotating fluid layer (see Chandrasekhar 1962, cha. III) is based upon the Oberbeck–Boussinesq approximation in which the thermal expansion of the fluid is neglected except where coupled with the gravitational acceleration, g. However, experiments by Koschmieder (1967) have suggested that in many circumstances the coupling of the thermal expansion and the centrifugal acceleration may play a significant role. Several theoretical studies have considered motions generated in convective systems by such centrifugal effects. Barcilon & Pedlosky (1967) considered the large-scale circulatory motions generated in a rapidly rotating cylinder when subject to a stable stratification, thus explicitly avoiding the onset of thermal instability. Homsy & Hudson (1969) extended this work to situations in which the centrifugal acceleration is large compared to that of gravity, and also to the case of unstable stratification (1971) where a stability analysis of the rapidly rotating centrifugally driven flow suggested an increase in the critical Rayleigh number due to the basic circulation, except possibly near the outer wall. Of course in this situation the centrifugal circulation necessarily dominates any circulation due to thermal instability. In the present paper we shall consider the situation in which the two motions are of comparable magnitude. An analysis of the thermal instability is necessarily restricted to the neighbourhood of the onset of the corresponding convective motions and it then emerges that the appropriate ratio of centrifugal and gravitational accelerations is given by a rotational Froude number

$$F = \Omega^2 Ld/g = O(L^{-1}). \tag{1.1}$$

Here Ω, d and Ld are respectively the angular velocity, height and outer radius of the annulus or cylinder and the basic assumption of our theory is that

$$L \gg 1. \tag{1.2}$$

Although we shall not consider cases in which overstability is the preferred mechanism of instability, the values of the Prandtl number and Taylor number defined in (2·8) below are otherwise completely arbitrary.

Because the horizontal scale of the motion due to the vertical instability is small compared to the horizontal scale of the container, the method of multiple scales may be used to expand the solution using L^{-1} as a small parameter, except in regions near the vertical curved walls of the annulus, or in the case of a cylinder, except near the outer wall and the centre. This expansion procedure follows the ideas of spatial modulation developed by Segel (1969) and Newell & Whitehead (1969) and more recently adapted to the type of cylindrical geometry with which we are concerned here by Brown & Stewartson (1978). They considered the non-rotating Bénard problem in a shallow cylinder and showed that such an expansion can be matched to consistent solutions both at the centre and at the outer curved wall of the cylinder. Our amplitude equation, determined in (4.1) below, may therefore be regarded as an extension to include both rotational and centrifugal effects although we shall see that the latter also result in an equally important contribution to the large-scale circulation in the cylinder.

The amplitude equation (4.1) may also be regarded as an extension of that obtained by the present author (Daniels 1978). There the basic model was equivalent to a rotating annulus in which both curvature and centrifugal effects were neglected, so that the width of the radial cross-section of the annulus must be taken as small compared to Ld and the value of F much less than that assumed in (1.1). One of the main purposse of the study, like that of Brown & Stewartson (1978), was to consider the effect of non-perfect insulation at the outer curved wall, which leads to a smooth transition to finite amplitude convection as the Rayleigh number is increased. The same effect could easily be incorporated in the present study but, as we shall see, in the rotating cylinder or annulus centrifugal effects characterized by (1.1) lead to a smooth transition even if the outer wall is perfectly insulated.

In § 2 we consider the modifications to the equations of motion caused by the non-Boussinesq assumption (1.1), and these are solved for the case of stress-free horizontal boundaries in § 3. Solutions of the resulting amplitude equation are discussed in § 4 and the modifications required for rigid boundaries in § 5. Further extensions of the theory to the case when just the upper surface is free are also discussed, and in § 6 a comparison is made with the experimental results of Koschmieder (1967).

2. The non-Boussinesq system

The equations of motion are

$$\frac{\partial \rho}{\partial t^*} + \nabla . (\rho \mathbf{u}^*) = 0,$$

$$\rho \left\{ \frac{\partial \mathbf{u}^*}{\partial t^*} + (\mathbf{u}^* . \nabla) \mathbf{u}^* \right\} = -\rho g \mathbf{k} - \nabla p^* + \mu \nabla^2 \mathbf{u}^* + \tfrac{1}{3}\mu \nabla (\nabla . \mathbf{u}^*),$$

$$\rho c_p \left\{ \frac{\partial \theta^*}{\partial t^*} + (\mathbf{u}^* . \nabla) \theta^* \right\} - \beta \theta^* \left\{ \frac{\partial p^*}{\partial t^*} + (\mathbf{u}^* . \nabla) p^* \right\} = \mu \nabla . (\mathbf{u}^* . \nabla \mathbf{u}^*) - \tfrac{2}{3}\mu (\nabla . \mathbf{u}^*)^2 + k \nabla^2 \theta^*, \tag{2.1}$$

where $\mathbf{u}^*, p^*, \theta^*$ and ρ are the velocity, pressure, temperature and density of the fluid, and $\beta = -\rho^{-1}(\partial\rho/\partial\theta^*)_{p^*}$ is the coefficient of thermal expansion. The specific heat at constant pressure, c_p, thermal conductivity, k, and coefficient of viscosity, μ, are taken to be constant and the bulk viscosity of the fluid is neglected; $\mathbf{k}$ is the vertical unit vector. The equation of state is taken as

$$\rho = \rho_0(1-\alpha_0[\theta^*-\theta_0^*]), \tag{2.2}$$

where α_0 is a constant and ρ_0 is the constant density at temperature θ_0^*, which is taken as the temperature of the lower horizontal surface of the container. The origin of a set of cylindrical polar co-ordinates (r^*, ϕ^*, z^*) is taken in this surface with z^* axis along the axis of rotation and an axisymmetric motion is assumed so that $\partial/\partial\phi^* = 0$. We now define non-dimensional dependent variables of velocity, temperature and pressure by (u, v, w), θ and p, where

$$\begin{aligned}
v^* &= \Omega r^* + \frac{\kappa}{d} v(r,z,t), \quad (u^*, w^*) = \frac{\kappa}{d}(u,w)(r,z,t),\\
\theta^* &= \theta_0^* + (\theta_1^* - \theta_0^*)\frac{z^*}{d} + \frac{\kappa\nu}{\alpha_0 g d^3}\theta(r,z,t),\\
p^* &= p_0^* - \rho_0 g z^* + \tfrac{1}{2}\rho_0 g\alpha_0(\theta_1^* - \theta_0^*)\frac{z^{*2}}{d} + \tfrac{1}{2}\rho_0\Omega^2 r^{*2} + \frac{\rho_0\kappa^2}{d^2}p(r,z,t).
\end{aligned} \tag{2.3}$$

Here p_0^* is a constant basic pressure, θ_1^* is the temperature of the upper surface of the container and $\nu = \mu/\rho_0$ and $\kappa = k/\rho_0 c_p$ are the kinematic viscosity and thermal diffusivity of the fluid at temperature θ_0^*; Ω is the angular velocity of the container and r, z, t are non-dimensional co-ordinates and times defined by

$$r = r^*/d, \quad z = z^*/d, \quad t = \frac{\kappa}{d^2}t^*. \tag{2.4}$$

The full set of equations (2.1), (2.2) may now be written in the form

$$\left.\begin{aligned}
\nabla.\mathbf{u} &= G\left\{\theta_t + \frac{1}{r}[ru(-Rz+\theta)]_r + [w(-Rz+\theta)]_z\right\},\\
\tilde{G}\left(u_t + uu_r + wu_z - \frac{v^2}{r} - \sigma T^{\frac{1}{2}}v\right) &= -p_r + \sigma\left\{\nabla^2 u - \frac{u}{r^2} + \frac{1}{3}\frac{\partial}{\partial r}(\nabla.\mathbf{u})\right\} - \tfrac{1}{4}r\sigma^2 TG(-Rz+\theta),\\
\tilde{G}\left(v_t + uv_r + wv_z + \frac{uv}{r} + \sigma T^{\frac{1}{2}}u\right) &= \sigma\left\{\nabla^2 v - \frac{v}{r^2}\right\},\\
\tilde{G}(w_t + uw_r + ww_z) &= -p_z + \sigma\left\{\nabla^2 w + \frac{1}{3}\frac{\partial}{\partial z}(\nabla.\mathbf{u})\right\} + \sigma\theta,\\
\tilde{G}(\theta_t + u\theta_r + w\theta_z - Rw) &= \nabla^2\theta + \epsilon,
\end{aligned}\right\} \tag{2.5}$$

where

$$\left.\begin{aligned}
\nabla^2 &= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2},\\
\nabla.\mathbf{u} &= \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z},\\
\tilde{G} &= 1 - G(-Rz+\theta),
\end{aligned}\right\} \tag{2.6}$$

and

$$\epsilon = \frac{\sigma H}{G}\left\{\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\left(uu_r+wu_z-\frac{v^2}{r}-\tfrac{1}{4}\sigma^2 Tr-\sigma T^{\frac{1}{2}}v\right)+\frac{\partial}{\partial z}(uw_r+ww_z)-\tfrac{2}{3}(\nabla.\mathbf{u})^2\right\}$$

$$+\frac{H}{G}(R_0-Rz+\theta)\{p_t+up_r+wp_z+\tfrac{1}{4}\sigma^2 rTu-\sigma w(G^{-1}+Rz)\}. \quad (2.7)$$

In this system of equations the Taylor number T, Prandtl number σ, Rayleigh number R, and three further parameters G, H and R_0 are defined by

$$\left.\begin{aligned} \mathrm{T} &= 4\Omega^2 d^4/\nu^2, & \sigma &= \nu/\kappa, & R &= \alpha_0 g d^3(\theta_0^*-\theta_1^*)/\kappa\nu, \\ G &= \kappa\nu/d^3 g, & H &= \kappa^2\alpha_0/c_p d^2, & R_0 &= \alpha_0 g d^3\theta_0^*\kappa/\nu. \end{aligned}\right\} \quad (2.8)$$

In these terms the Froude number F of (1.1). is $F = \frac{1}{4}\sigma TGL$, but we prefer to use G as the 'centrifugal parameter' of the system since the rotational effect characterized by the value of T then appears explicitly in the equations (2.5), the value of G being independent of the rate of rotation.

The Boussinesq approximation is equivalent to the assumption that both G and ϵ are negligible in (2.5) and (2.6) and in the non-rotating case this appears to be a good approximation in many situations, since both G and (especially) H/G are generally small (cf. Mihaljan 1962). In rotating systems, however, while it is likely that terms involving H are still negligible, the centrifugal term of $O(r\sigma^2 TG)$ in the radial momentum equation may be significant if, as in many experimental situations, either r or T is large, even though G itself is small. In the present theory, then, we shall assume that H is negligible, so that ϵ may be set to zero in (2.5) but shall choose the magnitude of G to be such that there is a significant interaction between the motions due to centrifugal effects and to the vertical instability, as appears to be the case in the experiments of Koschmieder (1967). Since our theory is to be based upon the assumption (1.2) we have $r = O(L)$ and it emerges (in § 3 below) that if the horizontal surfaces of the container are either both stress-free or both rigid we should set

$$G = L^{-2}G_0, \quad (2.09)$$

where $G_0 = O(1)$.

Initially we assume the horizontal surfaces to be stress free and maintained at the constant temperatures θ_0^* and θ_1^* so that

$$\theta = \partial u/\partial z = \partial v/\partial z = w = 0 \quad \text{at} \quad z = 0, 1. \quad (2.10)$$

To fix ideas we shall take an annular geometry with curved vertical walls which are rigid and perfectly insulated so that

$$\partial\theta/\partial r = u = v = w = 0 \quad \text{at} \quad r = s_0 L, L, \quad (2.11)$$

where $0 < s_0 < 1$. Of course for a complete cylinder the boundary conditions at the inner wall are not appropriate, although, as we shall see in § 3 below, the work of Brown & Stewartson (1978) suggests that, except near the centre of the cylinder, the solution will not be significantly different from that obtained from the solution for the annulus in the limit as $s_0 \to 0$.

It should be noted that the equation of state (2.2) is not strictly consistent with the assumption of a container of fixed volume since any increase in temperature with time will result in an overall expansion of the fluid, thereby violating the mass conservation condition

$$\frac{\partial}{\partial t^*}\left\{\int_{V^*} \rho \, dV^*\right\} = -\int_{S^*} \rho \mathbf{u}^* . \mathbf{dS}^* = 0, \tag{2.12}$$

obtained from the first of the equations (2.1), where V^* and S^* are the total volume and surface area of the container. However, although the coefficient of thermal expansion is not assumed to be negligible in the equation of continuity, its effect is too small to be significant here; any changes of the basic state with time may be assumed to be sufficiently slow to be incorporated in an insignificant seepage of fluid from the rim of the container.

3. Expansion procedure

Since the layer is shallow we may expect effects due to the vertical instability to become significant for Rayleigh numbers close to the critical value for the corresponding infinite layer. At values of σ and T for which the exchange of stabilities occurs, this critical Rayleigh number is a function of T alone, $R_c(T)$, given by

$$R_c = 3(\alpha^2 + \pi^2)^2, \tag{3.1}$$

where

$$\alpha^2 = (\tfrac{1}{2}\pi)^{\frac{2}{3}}\{(\tfrac{1}{2}\pi^4 + T + T^{\frac{1}{2}}[\pi^4 + T]^{\frac{1}{2}})^{\frac{1}{3}} + (\tfrac{1}{2}\pi^4 + T - T^{\frac{1}{2}}[\pi^4 + T]^{\frac{1}{2}})^{\frac{1}{3}}\} - \tfrac{1}{2}\pi^2. \tag{3.2}$$

Our analysis will provide solutions in the range

$$R = R_c(T) + L^{-2}\tilde{\delta} \quad (L \gg 1, \quad -\infty < \tilde{\delta} < \infty). \tag{3.3}$$

3.1. *Interior solution*

Away from the vertical walls of the annulus the solution may be expanded in the form

$$\begin{bmatrix} u \\ v \\ w \\ \theta \\ p \end{bmatrix}(r, z, t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \bar{p}_0(s) \end{bmatrix} + L^{-1}\begin{bmatrix} u_0 \\ v_0 \\ w_0 \\ \theta_0 \\ p_0 \end{bmatrix} + L^{-2}\begin{bmatrix} u_1 \\ v_1 \\ w_1 \\ \theta_1 \\ p_1 \end{bmatrix} + L^{-3}\begin{bmatrix} u_2 \\ v_2 \\ w_2 \\ \theta_2 \\ p_2 \end{bmatrix} + \ldots. \tag{3.4}$$

Here $u_i, v_i, w_i, \theta_i, p_i$ $(i = 0, 1, 2)$ are functions of r, z, s and $\tilde{\tau}$, where

$$s = L^{-1}r \quad (s_0 < s < 1), \quad \tilde{\tau} = L^{-2}t, \tag{3.5}$$

are slow variables of space and time appropriate to the dimensions of the container and the evolution of the motion for Rayleigh numbers (3.3) close to the critical value.

At order L^{-1} the solution for u_0 which satisfies the boundary conditions (2.10) consists of a cellular part which varies on the short length scale r and is characterized by an amplitude function, $A_0(s, \tilde{\tau})$, and a centrifugal part which is independent of the short length scale:

$$u_0 = (A_0 e^{i\alpha r} + A_0^* e^{-i\alpha r}) \cos \pi z + \tfrac{1}{4}s\sigma^2 T G_0 U_0(z). \tag{3.6}$$

The corresponding solutions for v_0, w_0, θ_0, p_0 and $\bar{p}_0$ are

$$\left.\begin{aligned}
v_0 &= -\frac{T^{\frac{1}{2}}}{\alpha^2+\pi^2}(A_0 e^{i\alpha r}+A_0^* e^{-i\alpha r})\cos\pi z+\tfrac{1}{4}s\sigma^2 TG_0 V_0(z),\\
w_0 &= -\frac{i\alpha}{\pi}(A_0 e^{i\alpha r}-A_0^* e^{-i\alpha r})\sin\pi z,\\
\theta_0 &= -\frac{i\alpha R_c}{\pi(\alpha^2+\pi^2)}(A_0 e^{i\alpha r}-A_0^* e^{-i\alpha r})\sin\pi z,\\
p_0 &= \frac{2i\alpha\sigma(\alpha^2+\pi^2)}{\pi^2}(A_0 e^{i\alpha r}-A_0^* e^{-i\alpha r})\cos\pi z+\bar{p}_1(s),\quad \bar{p}_0=\tfrac{1}{4}s^2\sigma^2 TG_0 P_0,
\end{aligned}\right\} \tag{3.7}$$

where P_0 is a constant, and $*$ denotes complex conjugate.

The centrifugal terms U_0, V_0 and P_0 satisfy the system of equations and boundary conditions

$$-\sigma T^{\frac{1}{2}}V_0 = -2P_0+\sigma U_0''+R_c z,\quad T^{\frac{1}{2}}U_0 = V_0''; \tag{3.8}$$

$$V_0' = U_0' = 0 \quad (z=0,\quad z=1).$$

This reduces to a fifth-order system for V_0 with four boundary conditions:

$$V_0^{\mathrm{v}}+TV_0' = -\sigma^{-1}R_c T^{\frac{1}{2}},\quad V_0' = V_0''' = 0 \quad (z=0,\quad z=1), \tag{3.9}$$

but we shall see in § 3.2 below that the fifth condition which uniquely determines the solution for V_0 is provided by the consistency of the solutions in the regions near the vertical walls of the annulus. This requires that V_0 be odd about $z=\frac{1}{2}$, and we then obtain

$$P_0 = \tfrac{1}{4}R_c,\quad V_0 = -\sigma^{-1}R_c T^{-\frac{1}{2}}(z-\tfrac{1}{2})+k_0(c_1\cosh z_1\sin z_1+c_2\sinh z_1\cos z_1), \tag{3.10}$$

where

$$z_1 = \frac{T^{\frac{1}{4}}}{\sqrt{2}}(z-\tfrac{1}{2}),\quad k_0 = \frac{R_c\sqrt{2}}{\sigma T^{\frac{3}{4}}(c_1^2+c_2^2)},\quad c_{1,2} = \cosh\zeta\cos\zeta\pm\sinh\zeta\sin\zeta,\quad \zeta = T^{\frac{1}{4}}/2\sqrt{2}. \tag{3.11}$$

The solution for U_0 is similar to that for V_0 apart from the linear term in z and the radial motion in the annulus consists of horizontal velocities, which increase linearly with s, in both inward and outward directions, reaching a maximum outward velocity at the upper surface $z=1$ and an equal and opposite inward velocity at the lower surface $z=0$, the centrifugal force on the cool heavy fluid at the top of the container being greater than that on the warmer, lighter fluid at the bottom. As $T\to\infty$, the radial motion is confined mainly to thin Ekman layers of thickness $O(T^{-\frac{1}{4}})$ along the two horizontal surfaces.

At order L^{-2} both the cellular and centrifugal terms in (3.6) and (3.7) produce nonlinear contributions in the equations of motion which suggest a solution for u_1 of the form

$$u_1 = (A_1 e^{i\alpha r}+A_1^* e^{-i\alpha r})\cos\pi z+\tfrac{1}{4}i\,(A_0 e^{i\alpha r}-A_0^* e^{-i\alpha r})\,\sigma^2 TG_0 s\bar{u}_1(z), \tag{3.12}$$

where $A_1(s,\tilde{\tau})$ is a further undetermined amplitude function. The curvature terms in the equations of motion (2.5) also enter the expansion at this stage, resulting in corresponding solutions for w_1, v_1 and θ_1 of the form

$$\begin{aligned}
w_1 &= (B_1 e^{i\alpha r} + B_1^* e^{-i\alpha r}) \sin \pi z + \tfrac{1}{4}(A_0 e^{i\alpha r} + A_0^* e^{-i\alpha r})\, \sigma^2 T G_0 s \bar{w}_1(z) + \tfrac{1}{4}\sigma^2 T G_0 W_1(z), \\
v_1 &= (C_1 e^{i\alpha r} + C_1^* e^{-i\alpha r}) \cos \pi z + \tfrac{1}{4} i (A_0 e^{i\alpha r} - A_0^* e^{-i\alpha r})\, \sigma^2 T G_0 s \bar{v}_1(z) \\
&\qquad + \frac{T^{\frac{1}{2}} i (A_0^2 e^{2i\alpha r} - A_0^{*2} e^{-2i\alpha r})}{4\sigma\alpha(\alpha^2+\pi^2)}, \\
\theta_1 &= (D_1 e^{i\alpha r} + D_1^* e^{-i\alpha r}) \sin \pi z + \tfrac{1}{4}(A_0 e^{i\alpha r} + A_0^* e^{-i\alpha r})\, \sigma^2 T G_0 s \bar{\theta}_1(z) \\
&\qquad + \tfrac{1}{4}\sigma^2 T G_0 \Theta_1(z) - \frac{\alpha^2 R_c |A_0|^2}{2\pi^3(\alpha^2+\pi^2)} \sin 2\pi z,
\end{aligned} \tag{3.13}$$

where

$$\begin{aligned}
B_1 &= -\frac{1}{\pi}\left(i\alpha A_1 + \frac{\partial A_0}{\partial s} + \frac{A_0}{s}\right), \quad C_1 = -\frac{T^{\frac{1}{2}}}{\alpha^2+\pi^2}\left(A_1 + \frac{i\alpha}{\alpha^2+\pi^2}\left[2\frac{\partial A_0}{\partial s} + \frac{A_0}{s}\right]\right), \\
D_1 &= -\frac{R_c}{\pi(\alpha^2+\pi^2)^2}\left(i\alpha[\alpha^2+\pi^2]A_1 + [\pi^2-\alpha^2]\frac{\partial A_0}{\partial s} + \pi^2\frac{A_0}{s}\right).
\end{aligned} \tag{3.14}$$

The equations for the centrifugal terms $\bar{u}_1$, $\bar{v}_1$, $\bar{w}_1$ and $\bar{\theta}_1$ may be reduced to a single sixth-order equation for $\bar{w}_1$:

$$\begin{aligned}
\bar{w}_1^{\text{vi}} - 3\alpha^2 \bar{w}_1^{\text{iv}} + (T + 3\alpha^4)\bar{w}_1'' + \alpha^2(R_c - \alpha^4)\bar{w}_1 \\
= \frac{\alpha^2}{\sigma}\{(b_0 + b_1 U_0' + b_2 U_0''')\cos \pi z + b_3 U_0 \sin \pi z\},
\end{aligned} \tag{3.15}$$

where

$$\begin{aligned}
b_0 &= 3(\sigma-1)(\alpha^2+\pi^2)^2/\sigma, \quad b_1 = (2\alpha^4 - \alpha^2\pi^2 - 3\pi^4)/\pi^2, \\
b_2 &= -3, \quad b_3 = (2T + 3\alpha^2\sigma(\alpha^2+\pi^2) + \alpha^2\pi^2 + 2\pi^4 - \alpha^4)/\pi
\end{aligned} \tag{3.16}$$

while the boundary conditions $\bar{w}_1 = \bar{u}_1' = \bar{v}_1' = \bar{\theta}_1 = 0$ at the horizontal surfaces become

$$\bar{w}_1 = \bar{w}_1'' = 0, \quad \bar{w}_1^{\text{iv}}(0) = -\bar{w}_1^{\text{iv}}(1) = -\frac{\alpha^2 R_c}{\sigma(\alpha^2+\pi^2)}. \tag{3.17}$$

The solution for $\bar{w}_1$ is

$$\begin{aligned}
\bar{w}_1 &= a_0 \cos \pi z + a_1 \cosh k_1(z-\tfrac{1}{2}) \sin k_2(z-\tfrac{1}{2}) + a_2 \sinh k_1(z-\tfrac{1}{2}) \cos k_2(z-\tfrac{1}{2}) \\
&\quad + e_0(z-\tfrac{1}{2})\sin \pi z + (e_1 U_0 + e_2 U_0'')\sin \pi z + (e_3 U_0' + e_4 U_0''')\cos \pi z,
\end{aligned} \tag{3.18}$$

where $k = \pm(k_1 \pm ik_2)$ are the four roots of the equation

$$k^6 - 3\alpha^2 k^4 + (3\alpha^4 + T)k^2 + \alpha^2(R_c - \alpha^4) = 0, \tag{3.19}$$

which in conjunction with the other two roots, $\pm i\pi$, give rise to the three complementary solutions associated with the constants a_0, a_1 and a_2. Note that since the boundary conditions (3.17) and right-hand side of (3.15) are odd about $z = \frac{1}{2}$, $\bar{w}_1$ must also be odd. Indeed, if either had contained components which were even about $z = \frac{1}{2}$ an inconsistency would arise since one of the even complementary solutions of (3.15) for $\bar{w}_1$ is $\sin \pi z$ and so only five constants would be available to satisfy the six conditions (3.17). Had this been the case, centrifugal effects on the cellular motion would be more pronounced and the appropriate choice of the magnitude of G would have been $O(L^{-3})$

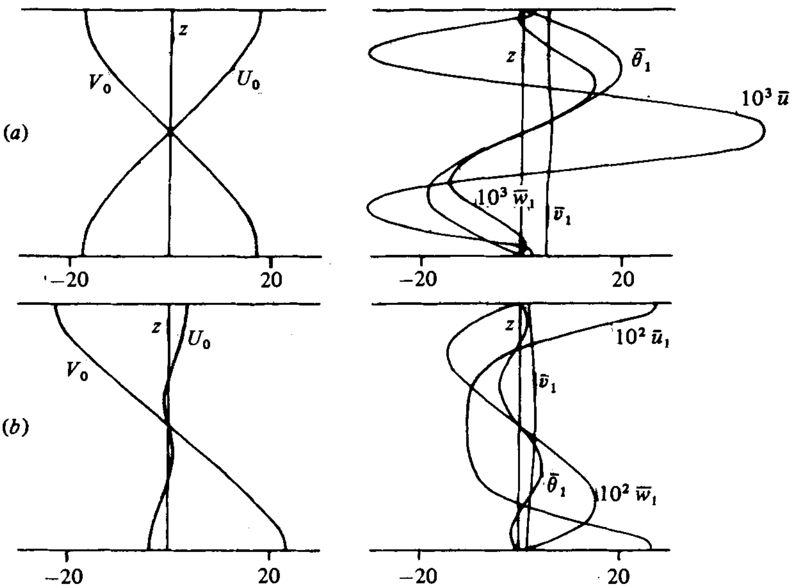

FIGURE 1. Typical profiles of the centrifugal functions U_0, V_0 and $\bar u_1$, $\bar w_1$, $\bar v_1$, $\bar\theta_1$ for free-free boundaries: (a) $\sigma = 1$, $T = 10^2$, (b) $\sigma = 1$, $T = 10^4$.

in place of (2.9). The particular solutions associated with the constants $e_0, \ldots, e_4$ (which may be determined in terms of $b_0, \ldots, b_3$) may be restricted to at most third-order derivatives of U_0 since higher-order derivatives may be replaced using the relation $U_0^{\mathrm{iv}} = -TU_0$.

Having found $\bar w_1$, the solutions for $\bar u_1$ and $\bar v_1$ (which are even about $z = \frac{1}{2}$) are determined from

$$-\alpha\bar u_1 + \bar w_1' = 0,$$

$$\sigma(\alpha^2\bar v_1 - \bar v_1'') = -\sigma T^{\frac{1}{2}}\bar u_1 + \alpha\left(\frac{T^{\frac{1}{2}}}{\alpha^2+\pi^2}U_0\cos\pi z + \frac{V_0'}{\pi}\sin\pi z\right), \tag{3.20}$$

in turn, using the boundary conditions $\bar v_1' = 0$ at $z = 0, 1$ to fix the coefficient of the complementary function $\cosh\alpha(z-\frac{1}{2})$ in the solution for $\bar v_1$. A corresponding term $\sinh\alpha(z-\frac{1}{2})$ occurs in the solution for $\bar\theta_1$, which is odd about $z = \frac{1}{2}$ and may be determined from

$$\alpha^2\bar\theta_1 - \bar\theta_1'' = R_c\bar w_1 - \frac{\alpha^2 R_c U_0}{\pi(\alpha^2+\pi^2)}\sin\pi z, \quad \bar\theta_1(0) = \bar\theta_1(1) = 0. \tag{3.21}$$

The solutions for the centrifugal terms W_1 and Θ_1 are found to be

$$W_1 = -2\int_0^z U_0\,dz, \quad \Theta_1 = \left(zR_c\int_0^1 - R_c\int_0^z\right)\left\{\int_0^{z'} W_1(z'')\,dz''\right\}dz'. \tag{3.22}$$

The solution for W_1 automatically vanishes at $z = 1$ since U_0 is odd about $z = \frac{1}{2}$, a reflexion of the requirement that the total flux across any vertical line in the vertical cross-section of the annulus must be zero.

At order L^{-3} the solution for w_2 contains a component $e^{i\alpha r}\overline{w}_2(z, s, \tilde{\tau})$ and $\overline{w}_2$ may be shown to satisfy an equation similar to that (3.15) for $\overline{w}_1$, but with a right-hand side which now contains both even and odd components which include effects resulting from curvature, nonlinear interactions and variations with $\tilde{\tau}$. It may be shown that

$$\left.\frac{\partial^4 \overline{w}_2}{\partial z^4}\right|_{z=0} + \left.\frac{\partial^4 \overline{w}_2}{\partial z^4}\right|_{z=1} = -\tfrac{1}{8}\alpha i \sigma^3 T^2 G_0^2 s^2 A_0 \overline{\theta}_1'(0), \quad \left.\frac{\partial^2 \overline{w}_2}{\partial z^2}\right|_{z=0} + \left.\frac{\partial^2 \overline{w}_2}{\partial z^2}\right|_{z=1} = 0,$$

in addition to $\overline{w}_2 = 0$ at $z = 0, 1$ so that multiplication of both sides of the equation by $\sin \pi z$ and integration from $z = 0$ to $z = 1$ yields a condition on the terms which appear on the right-hand side. Only the even components contribute to this condition and we finally obtain

$$\left\{\frac{\alpha^2+\pi^2}{\pi}(2\pi^2+\alpha^2[3\sigma-1])\right\}\frac{\partial A_0}{\partial \tilde{\tau}} - \frac{12\sigma\alpha^2}{\pi}(\alpha^2+\pi^2)\left\{\frac{\partial^2 A_0}{\partial s^2} + \frac{1}{s}\frac{\partial A_0}{\partial s} - \frac{A_0}{4s^2}\right\}$$
$$-\frac{\sigma}{\pi}\{\alpha^2\tilde{\delta} + G_0 R_c(\tfrac{1}{2}R_c\alpha^2 - T\pi^2)\}A_0 + \frac{\alpha^2+\pi^2}{2\sigma\pi^3}\{3\alpha^4\sigma^2 - \pi^2(2\alpha^2-\pi^2)\}A_0|A_0|^2$$
$$-\tfrac{1}{8}\sigma^4 T^2 G_0^2 \Gamma s^2 A_0 = 0, \tag{3.23}$$

where

$$\Gamma(\sigma, T) = (\alpha^4-\pi^4)\int_0^1 \overline{w}_1 U_0 \sin \pi z\, dz - \pi^2 T^{\frac{1}{2}}\int_0^1 \overline{w}_1 V_0 \sin \pi z\, dz + \alpha^2\sigma\int_0^1 \overline{\theta}_1 U_0 \sin \pi z\, dz$$
$$+ 2\alpha\pi(\alpha^2+\pi^2)\int_0^1 \overline{u}_1 U_0 \cos \pi z\, dz + T^{\frac{1}{2}}\alpha\pi\int_0^1 (\overline{u}_1 V_0 + \overline{v}_1 U_0)\cos \pi z\, dz$$
$$-\pi(\alpha^2+\pi^2)\int_0^1 \overline{\theta}_1 \cos \pi z\, dz. \tag{3.24}$$

Products of the centrifugal terms U_0, V_0 with $\overline{u}_1$, $\overline{v}_1$, $\overline{w}_1$ and $\overline{\theta}_1$ in the nonlinear terms in the equations and the centrifugal term in θ_1 both contribute to the final term in the amplitude equation, whose dependence on s^2 reflects the fact that centrifugal effects are strongest in the outer parts of the annulus. The contribution from G_0 in the non-centrifugal terms in the equations (2.5) also results in a shift in the effective value of the Rayleigh number (characterized by $\tilde{\delta}$) – a correction due to the slightly non-Boussinesq nature of the fluid.

3.2. *The side-wall solution*

The interior solution obtained above must be matched to consistent solutions near each curved wall of the annulus in order to obtain the boundary conditions for $A_0(s, \tilde{\tau})$. We therefore fix attention on the region near the outer wall where

$$r_1 = r - L = O(1). \tag{3.25}$$

Nonlinear effects are not significant here and some general properties of the linear equations are relevant. If $u \sim L^{-1}\overline{u}(r_1, z)$, $v \sim L^{-1}\overline{v}(r_1, z)$ then from the second equation of (2.5) we have

$$\sigma T^{\frac{1}{2}}\int_0^1 \overline{u}\, dz = \sigma\left\{\frac{\partial^2}{\partial r_1^2}\int_0^1 \overline{v}\, dz + \frac{\partial \overline{v}}{\partial z}(r_1, 1) - \frac{\partial \overline{v}}{\partial z}(r_1, 0)\right\}. \tag{3.26}$$

But flux considerations imply that the left-hand side is zero and since the upper and lower surfaces are stress-free we obtain

$$\int_0^1 \bar{v}(r_1, z)\,dz = K_1 r_1 + K_2, \tag{3.27}$$

where K_1 and K_2 are constants. But $\bar{v} = 0$ on $r_1 = 0$ so that $K_2 = 0$ and provided that $\bar{v} = O(1)$ as $r_1 \to \infty$ we also have $K_1 = 0$. Thus

$$\int_0^1 \bar{v}(r_1, z)\,dz = 0, \tag{3.28}$$

and, in particular, taking the limit as $r_1 \to \infty$ and matching with the interior solution (3.7) implies that

$$\int_0^1 V_0(z)\,dz = 0. \tag{3.29}$$

This is the fifth condition which uniquely determines the solution of the centrifugal system (3.9) and implies that both U_0 and V_0 are odd about $z = \frac{1}{2}$.

We find that the explicit solution for u in the end region has the form

$$u = L^{-1}\left\{(A_0(1,\tilde{\tau})\,e^{i\alpha r} + A_0^*(1,\tilde{\tau})\,e^{-i\alpha r} + E_1 e^{\alpha_1 r_1})\cos \pi z + \sum_{n=2}^{\infty} H_n(r_1)\cos n\pi z + \tfrac{1}{4}\sigma^2 T G_0 U_0(z)\right\} + O(L^{-2}), \tag{3.30}$$

where $\alpha_1 = (3\pi^2 + 2\alpha^2)^{\frac{1}{2}}$ and E_1 is a real constant. The functions $H_n(r_1)$ $(n = 2, 3, \ldots)$ are each linear combinations of three exponential solutions which decay as $r_1 \to -\infty$, (see Daniels 1978). The solution (3.30) then matches with the interior solution (3.6) as $r_1 \to -\infty$, and satisfies the boundary condition $u = 0$ at the wall $r_1 = 0$ provided that

$$(A_0(1,\tilde{\tau})\,e^{i\alpha L} + A_0^*(1,\tilde{\tau})\,e^{-i\alpha L} + E_1)\cos \pi z + \sum_{n=2}^{\infty} H_n(0)\cos n\pi z = -\tfrac{1}{4}G_0\sigma^2 T U_0(z). \tag{3.31}$$

The end-region solutions for v, w and θ are expressed in a similar form to that for u although those for θ and v contain a further set of arbitrary constants F_n $(n = 1, 2, \ldots)$ (see Daniels 1978). Satisfaction of the remaining three boundary conditions in (2.11) then leads to three more equations to be solved in conjunction with (3.31) for each set of four unknown constants. The first Fourier components of U_0 and V_0 determine the leading set $(n = 1)$ as the solutions X_1, X_2, E_1, F_1 of the system

$$\begin{gathered} X_1 + E_1 = -\tfrac{1}{2}G_0\sigma^2 T \int_0^1 U_0 \cos \pi z\,dz, \\ \alpha X_2 + \alpha_1 E_1 = 0, \\ \frac{T^{\frac{1}{2}}}{\alpha^2+\pi^2}(X_1 - \tfrac{1}{2}E_1) + T^{-\frac{1}{2}}F_1 = \tfrac{1}{2}G_0\sigma^2 T \int_0^1 V_0 \cos \pi z\,dz, \\ \frac{R_c}{\pi(\alpha^2+\pi^2)}(\alpha^2 X_1 + \tfrac{1}{2}\alpha_1^2 E_1) + \pi F_1 = 0, \end{gathered} \tag{3.32}$$

where

$$X_1 = A_0(1,\tilde{\tau})\,e^{i\alpha L} + A_0^*(1,\tilde{\tau})\,e^{-i\alpha L}, \quad X_2 = i(A_0(1,\tilde{\tau})\,e^{i\alpha L} - A_0^*(1,\tilde{\tau})\,e^{-i\alpha L}). \tag{3.33}$$

From the properties of U_0 and V_0 and the resulting solutions for X_1 and X_2 we obtain

$$A_0(1,\tilde{\tau}) = \frac{\sigma T G_0 \alpha_1 (\pi^2 + 2\alpha^2)^{\frac{1}{2}} R_c}{2\sqrt{3}\,(\pi^4 + T)\,\pi^2}\,e^{-i\beta_0}, \tag{3.34}$$

where

$$\beta_0 = \alpha L + \tan^{-1}\left(\frac{2\alpha}{\alpha_1}\right). \tag{3.35}$$

Thus the second and most striking effect of the centrifugal acceleration is to provide a non-zero boundary condition for the cellular amplitude function at the outer wall $s = 1$. For the annulus a similar condition will provide the value of $A_0(s_0, \tilde{\tau})$ to complete the boundary conditions for the solution of (3.23), the major difference from (3.34) being a reduction in the right-hand side by the factor s_0. Thus if we consider the limit $s_0 \to 0$, the inner boundary condition on A_0 is

$$A_0(0, \tilde{\tau}) = 0 \tag{3.36}$$

to leading order. Of course this argument does not allow us to consider the corresponding solution as that for a complete cylinder, since once $s_0 = O(L^{-1})$ the curvature effects near the centre of the container will appear in the equations at leading order and must be taken into account. However, Brown & Stewartson (1978) have considered this problem in the non-rotating case and their results strongly suggest that, even if the central section is included in the flow field, the boundary condition (3.36) is still appropriate, although a closer examination of the behaviour of A_0 near $s = 0$ (Brown & Stewartson 1979) is required to detect the precise nature of the relatively weak motion which occurs at the onset of instability.

4. Solutions of the amplitude equation

The equation (3.23) and boundary conditions (3.34) and (3.36) may be reduced to the form

$$\frac{\partial A}{\partial \tau} = \frac{\partial^2 A}{\partial s^2} + \delta A + s^2 \gamma A - \frac{1}{s} A\,|A|^2 \,\mathrm{sgn}\,(\xi),$$

$$A(0) = 0, \quad A(1) = \lambda, \tag{4.1}$$

where

$$A(s, \tau) = s^{\frac{1}{2}} |\xi|^{\frac{1}{2}} A_0(s, \tilde{\tau})\, e^{i\beta_0}, \quad \tau = \frac{12\sigma\alpha^2\tilde{\tau}}{2\pi^2 + \alpha^2(3\sigma - 1)}, \tag{4.2}$$

and

$$\delta = \frac{1}{12(\alpha^2+\pi^2)}\left\{\tilde{\delta} + \frac{G_0 R_c}{\alpha^2}\left(\tfrac{1}{2}R_c\alpha^2 - T\pi^2\right)\right\}, \tag{4.3}$$

$$\gamma = \frac{\pi\sigma^3 T^2 G_0^2\, \Gamma(\sigma, T)}{96\alpha^2(\alpha^2+\pi^2)}, \tag{4.4}$$

$$\xi = \frac{3\alpha^4\sigma^2 - \pi^2(2\alpha^2 - \pi^2)}{24\sigma^2\pi^2\alpha^2}, \tag{4.5}$$

$$\lambda = \frac{\sigma T G_0 R_c (\pi^2 + 2\alpha^2)^{\frac{1}{2}} (3\pi^2 + 2\alpha^2)^{\frac{1}{2}} |\xi|^{\frac{1}{2}}}{(\pi^4 + T)\, 2\sqrt{3}\, \pi^2}. \tag{4.6}$$

The transformation from A_0 to A effectively removes the curvature terms from the equation, as in the non-rotating study of Brown & Stewartson (1978). The constant ξ is unchanged from its two-dimensional value (see Daniels 1978) and can be either

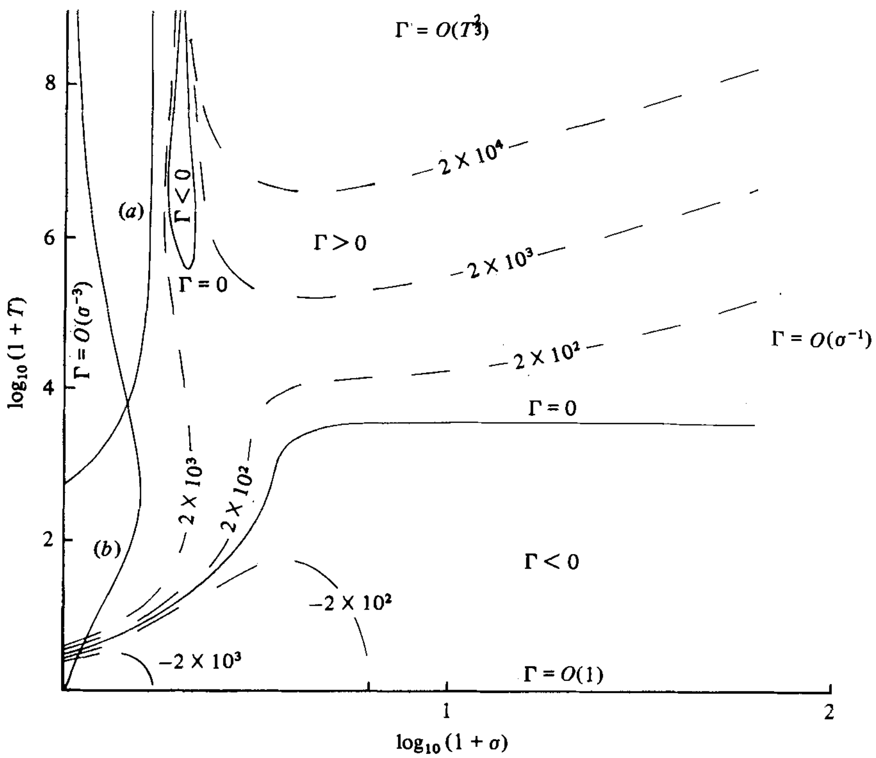

FIGURE 2. Contours of the centrifugal parameter Γ as a function of σ and T. Also shown are (i) the region in which overstability is preferred to the exchange of stabilities (to the left of curve (*a*), taken from Veronis 1959) and (ii) the region of subcritical instability ($\xi < 0$) (to the left of curve (*b*), taken from Daniels 1978).

positive or negative, suggesting that both supercritical and subcritical behaviour will be possible. The sign of the centrifugal parameter γ is the same as that of $\Gamma(\sigma, T)$ given by (3.24) and, as shown in figure 2, can also be positive or negative, depending on the values of σ and T. The shift in the value of δ in (4.3) due to G_0 is positive for $T < 2360$ and negative for $T > 2360$.

The nature of the solution of (4.1) will depend upon the value of each of the four independent parameters $\tilde{\delta}$, σ, T and G_0 which determine the values of δ, γ, ξ and λ through the transformations (4.3)–(4.6). The effect of the parameter λ has already been discussed in previous studies of related systems (Daniels 1977, 1978; Brown & Stewartson 1978) where its non-zero value represents an imperfect insulation of the side walls of the container; the bifurcation of the solution for A at a critical value of δ when $\lambda = 0$ is replaced by a smooth transition as the Rayleigh number increases. Here the size of λ is fixed by (4.6) and, provided that G_0, T and ξ are non-zero, λ is strictly positive.

4.1. *Solutions for $G_0 \ll 1$*

If $G_0 \ll 1$ we may write $\lambda = \lambda_0 G_0$, where $\lambda_0 = O(1)$ and the steady solution of (4.1) develops smoothly from the linear solution

$$A = G_0 \lambda_0 \sin \delta^{\frac{1}{2}} s / \sin \delta^{\frac{1}{2}} \quad (0 < \delta < \pi^2), \tag{4.7}$$

into the nonlinear form

$$A = G_0^{\frac{1}{3}} \lambda_0^{\frac{1}{3}} \bar{a}_0 \sin \pi s + O(G_0), \tag{4.8}$$

in the region $y_0 = \lambda_0^{-\frac{2}{3}} G_0^{-\frac{2}{3}} (\delta - \pi^2) = O(1)$, where $\bar{a}_0$ is the solution of

$$\operatorname{sgn}(\xi) p_0 \bar{a}_0^3 - \tfrac{1}{2} y_0 \bar{a}_0 - \pi = 0 \quad (p_0 = 0{\cdot}830\ldots), \tag{4.9}$$

which matches with (4.7) as $y_0 \to -\infty$. Note that the motion (4.8) due to the vertical instability now dominates the centrifugal circulation, which is $O(G_0)$. If $\xi > 0$ the value of $\bar{a}_0$ continues to increase as $y_0 \to \infty$ and (4.8) develops into a full nonlinear solution of the system (4.1) for $\delta > \pi^2$, eventually attaining the limiting form

$$A \sim \delta^{\frac{1}{2}} s^{\frac{1}{2}} \quad (0 < s < 1), \tag{4.10}$$

as $\delta \to \infty$, representing convection cells of uniform strength across the width of the container. Other real solutions of (4.1) are found to be unstable, details of the stability analysis, evolution properties and derivation of (4.9) for $\xi > 0$ being given by Brown & Stewartson (1978). If $\xi < 0$, the solution of (4.9) for $\bar{a}_0$ which evolves from (4.7) no longer continues to increase with y_0 but reaches a turning point at $y_0 = -(54\pi^2 p_0)^{\frac{1}{3}}$, where it becomes unstable and matches as $y_0 \to -\infty$ with an unstable nonlinear solution of (4.1) in the region $\delta < \pi^2$. This takes the limiting form

$$A \sim (-2\delta)^{\frac{1}{2}} s^{\frac{1}{2}} \operatorname{sech} (-\delta)^{\frac{1}{2}} s \quad (0 < s < 1), \tag{4.11}$$

as $\delta \to -\infty$. This type of subcritical behaviour and the associated phenomenon of bursting is discussed in greater detail by Daniels (1978).

4.2. *Solutions for $T \gg 1$*

We have seen that, with $G_0 \ll 1$ and finite T, centrifugal effects appear mainly through the parameter λ, evoking an early appearance of convection cells at the outer edge of the container as the Rayleigh number is increased. Since $\gamma = O(G_0^2)$ its effect in the equation (4.1) is of little significance in this situation. However, with $G_0 = O(1)$ or even with $G_0 \ll 1$ if $T \gg 1$, the parameter γ may play a significant role in (4.1). In general, depending on whether Γ is positive or negative, its effect will be to either diminish or increase the effective value of the scaled Rayleigh number, δ, with a strength proportional to the square of the local radius. To see how this can occur we consider the limiting situation as $T \to \infty$. The centrifugal circulation is then mainly confined to thin boundary layers around the walls of the container. We have

$$\alpha \sim (\tfrac{1}{2}\pi^2 T)^{\frac{1}{6}}, \quad R_c \sim 3(\tfrac{1}{2}\pi^2 T)^{\frac{2}{3}}, \quad \Gamma \sim \frac{9\pi}{16} (\tfrac{1}{2}\pi^2 T)^{\frac{2}{3}} \frac{(1-\sigma)^2}{\sigma^3}, \tag{4.12}$$

so that $\xi > 0$ andf except near $\sigma = 1$, $\gamma > 0$. We may write

$$\gamma = \gamma_1 T^2, \quad \lambda = \lambda_1 T^{\frac{7}{6}}, \tag{4.13}$$

where $\gamma_1, \lambda_1 = O(1)$. Significant changes in the solution of the amplitude equation then occur for Rayleigh numbers in the range

$$\delta = T^2 \delta_1, \quad \delta_1 = O(1), \tag{4.14}$$

when the amplitude of the motion is

$$A = TA_1, \tag{4.15}$$

where A_1 is order one and, for steady solutions, satisfies the equation

$$\frac{\partial^2 A_1}{\partial s^2} + T^2[(\delta_1 + s^2\gamma_1)A_1 - \frac{1}{s}A_1^3] = 0, \tag{4.16}$$

with boundary conditions

$$A_1(0) = 0, \quad A_1(1) = T^{\frac{1}{3}}\lambda_1. \tag{4.17}$$

Although the size of λ suggests a different scaling from (4.15), as evidenced by the second condition of (4.17), its effect is limited to a boundary layer near the outer wall, where $s_1 = T^{\frac{2}{3}}(s-1) = O(1)$. Here the first and last terms of (4.16) balance and

$$A_1 \approx T^{\frac{1}{3}}\left(\frac{\sqrt{2}\,\lambda_1}{\sqrt{2}-\lambda_1 s_1}\right) \quad (-\infty < s_1 < 0), \tag{4.18}$$

so that $A_1 \sim -T^{\frac{1}{3}}\sqrt{2}\, s_1^{-1}$ as $s_1 \to -\infty$.

The solution of (4.16) where $s = O(1)$ is crucially dependent upon the values of δ_1 and γ_1; we take $\gamma_1 > 0$. For $\delta_1 < -\gamma_1$ the solution for A_1 where $s = O(1)$ is essentially

$$A_1(s) = 0 \tag{4.19}$$

and at $s = 1$ this adjusts to (4.18) through a boundary layer where $s_2 = T(s-1) = O(1)$. Here $A_1 \approx \bar{A}_1(s_2)$ and

$$\frac{d^2\bar{A}_1}{ds_2^2} + (\delta_1 + \gamma_1)\bar{A}_1 - \bar{A}_1^3 = 0, \tag{4.20}$$

and matching requires

$$\bar{A}_1 \to 0 \quad (s_2 \to -\infty), \quad \bar{A}_1 \sim -\sqrt{2}\, s_2^{-1} \quad (s_2 \to 0). \tag{4.21}$$

The solution is

$$\bar{A}_1 = \frac{2\sqrt{2}\,(-\delta_1-\gamma_1)^{\frac{1}{2}}\, e^{(-\delta_1-\gamma_1)^{\frac{1}{2}} s_2}}{1 - e^{2(-\delta_1-\gamma_1)^{\frac{1}{2}} s_2}}. \tag{4.22}$$

For $-\gamma_1 < \delta_1 < 0$ the solution (4.19) still holds inside the critical radius

$$s = (-\delta_1/\gamma_1)^{\frac{1}{2}} \tag{4.23}$$

but outside we expect the stable solution of (4.16) to be

$$A_1 \approx s^{\frac{1}{2}}(\delta_1 + \gamma_1 s^2)^{\frac{1}{2}}, \tag{4.24}$$

so that the amplitude of the cells now increases from zero at the value of s given by (4.23) to a maximum value at $s = 1$. Here the solution again adjusts to (4.18) through the boundary layer where $s_2 = O(1)$ but the appropriate boundary conditions for $\bar{A}_1$ are now

$$\bar{A}_1 \to (\delta_1 + \gamma_1)^{\frac{1}{2}} \quad (s_2 \to -\infty), \quad \bar{A}_1 \sim -\sqrt{2}\, s_2^{-1} \quad (s_2 \to 0), \tag{4.25}$$

and the solution of (4.20) is

$$\bar{A}_1 = -(\delta_1 + \gamma_1)^{\frac{1}{2}} \coth\left\{\frac{(\delta_1+\gamma_1)^{\frac{1}{2}} s_2}{\sqrt{2}}\right\}. \tag{4.26}$$

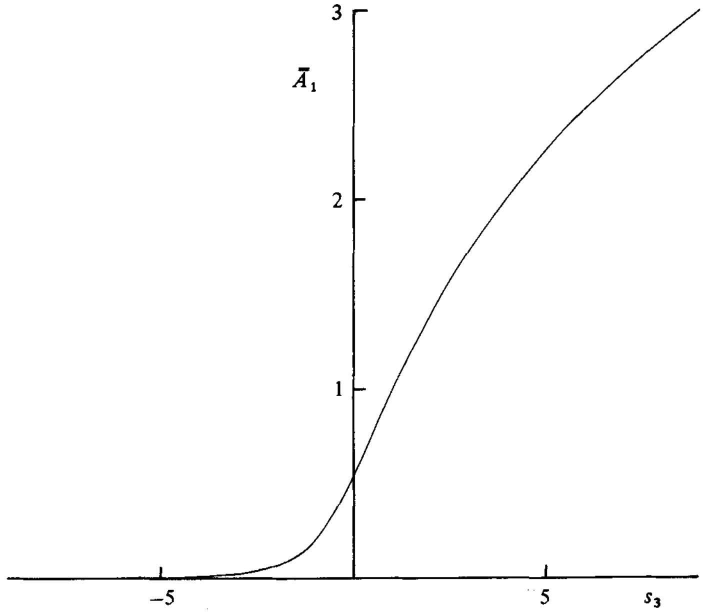

FIGURE 3. The solution of the nonlinear Airy equation (4.28) with boundary conditions (4.29).

In the neighbourhood of the critical radius given by (4.23) the transition from (4.19) to (4.24) occurs in the region where

$$\left.\begin{aligned} s &= (-\delta_1/\gamma_1)^{\frac{1}{2}} + T^{-\frac{2}{3}}2^{-\frac{1}{3}}\gamma_1^{-\frac{1}{6}}(-\delta_1)^{-\frac{1}{6}}s_3, \\ A_1 &\approx T^{-\frac{1}{3}}2^{\frac{1}{3}}\gamma_1^{-\frac{1}{12}}(-\delta_1)^{\frac{5}{12}}\bar{\bar{A}}_1(s_3), \end{aligned}\right\} \tag{4.27}$$

and

$$\frac{d^2\bar{\bar{A}}_1}{ds_3^2} + s_3\bar{\bar{A}}_1 - \bar{\bar{A}}_1^3 = 0. \tag{4.28}$$

Matching with (4.19) and (4.24) implies that

$$\bar{\bar{A}}_1 = O(-s_3)^{-\frac{1}{4}}\exp\left(-\tfrac{2}{3}(-s_3)^{\frac{3}{2}}\right) \quad (s_3 \to -\infty), \quad \bar{\bar{A}}_1 \sim s_3^{\frac{1}{2}} \quad (s_3 \to \infty), \tag{4.29}$$

and the solution was found numerically by computing forward from $s_3 = s_{30}$ (taken as -10) by applying a small positive increment to the value of $\bar{\bar{A}}_1$. Too large an increment results in the development of the singular solution $\bar{\bar{A}}_1 \sim 2\,(s_{30} - s_3)^{-1}$ at some point s_{30}, while too small an increment leads to the evolution of the asymptotic Airy function solutions as $s_3 \to \infty$. The correct value is thus obtained iteratively and leads to the required behaviour (4.29), with $\bar{\bar{A}}_1(0) = 0{\cdot}52\ldots$, $\bar{\bar{A}}_1'(0) = 0{\cdot}42\ldots$ (see figure 3).

At $\delta_1 = 0$ the critical radius reaches the inner limit $s = 0$ and for $\delta_1 > 0$ the interior solution (4.24) applies for all $0 < s < 1$. At the outer wall (4.26) and (4.18) are still appropriate while at $s = 0$ there is a further region where $s = O(T^{-1})$ and only the centrifugal term in (4.16) may be neglected. The solution here is the same as that found numerically by Brown & Stewartson (1978, equation (5.13)).

We conclude that if T is large so that, except for $\sigma \simeq 1$, γ is positive, the centrifugal term in equation (4.1) leads to a decrease in the value of the Rayleigh number at which cellular convection sets in. The amplitude of the motion increases with radius and initially (for $\delta_1 < 0$) is restricted to the region outside the critical radius (4.23). As the Rayleigh number increases, this radius decreases to zero and for $\delta_1 > 0$ cellular convection occurs throughout the container, initially increasing in strength with radius, but ultimately (as $\delta_1 \to \infty$) attaining the uniform distribution (4.10).

Although the region of parameter space in which $\gamma < 0$ is restricted to the neighbourhood of $\sigma = 1$ when T is large (see figure 2), a similar solution structure to that outlined above may be envisaged. The main differences are that, except for the side-wall effect (4.18), cellular convection is now initially restricted to the region inside the critical radius $s = (\delta_1/-\gamma_1)^{\frac{1}{2}}$ and first occurs at much higher Rayleigh numbers ($\delta_1 > 0$). As $\delta_1 \to -\gamma_1$ the critical radius reaches the outer wall, (4.24) is then valid for all $0 < s < 1$ and the uniform distribution (4.10) is again attained as $\delta_1 \to \infty$.

5. Rigid horizontal surfaces

In this section we consider the modifications to the analysis of §§ 3 and 4 resulting from the replacement of (2.10) by the more realistic conditions

$$u = v = w = \theta = 0 \quad \text{at} \quad z = 0, 1, \tag{5.1}$$

appropriate to rigid horizontal surfaces. The vertical eigenfunctions are no longer sinusoidal and we must now assume interior solutions of the form

$$\begin{aligned} u_0 &= (A_0 e^{i\alpha r} + A_0^* e^{-i\alpha r})\,\Phi'(z) + \tfrac{1}{4}s\sigma^2 T G_0 U_0(z), \\ v_0 &= (A_0 e^{i\alpha r} + A_0^* e^{-i\alpha r})\,V(z) + \tfrac{1}{4}s\sigma^2 T G_0 V_0(z), \\ \theta_0 &= i(A_0 e^{i\alpha r} - A_0^* e^{-i\alpha r})\,\Theta(z). \end{aligned} \tag{5.2}$$

The functions Φ, Θ and V satisfy

$$\left.\begin{aligned} &\Theta'' - \alpha^2\Theta - R_c\alpha\Phi = 0, \\ &V'' - \alpha^2 V - T^{\frac{1}{2}}\Phi' = 0, \\ &\Phi^{\text{iv}} - 2\alpha^2\Phi'' + \alpha^4\Phi + \alpha\Theta + T^{\frac{1}{2}}V' = 0, \end{aligned}\right\} \tag{5.3}$$

and the critical wavenumber α and Rayleigh number R_c must be determined from the solution of these equations which satisfies the conditions

$$\Theta = V = \Phi = \Phi' = 0 \quad (z = 0, 1), \tag{5.4}$$

and at which $dR/d\alpha = 0$. It may be shown from (5.3) that the latter condition is equivalent to

$$I \equiv \int_0^1 \left(\frac{2\alpha}{R_c}\Theta^2 + 2\Theta\Phi - 2\alpha V^2 - 4\alpha\Phi\Phi'' + 4\alpha^3\Phi^2\right) dz = 0. \tag{5.5}$$

It is clear that V will be odd about $z = \frac{1}{2}$ while Φ and Θ will be even.

In the centrifugal system (3.9) the boundary conditions now become $V_0 = V_0'' = 0$ at $z = 0, 1$ and it no longer follows from (3.26) that $\int_0^1 V_0\,dz = 0$. However, flux considerations require that $\int_0^1 U_0\,dz = 0$ and from the second equation of (3.8) we obtain $V_0'(0) = V_0'(1)$. Thus the fifth-order system for V_0 is again completed and a solution similar to (3.10) can be constructed. As in the free case, both U_0 and V_0 are odd about $z = \frac{1}{2}$ and some profiles are shown in figure 4.

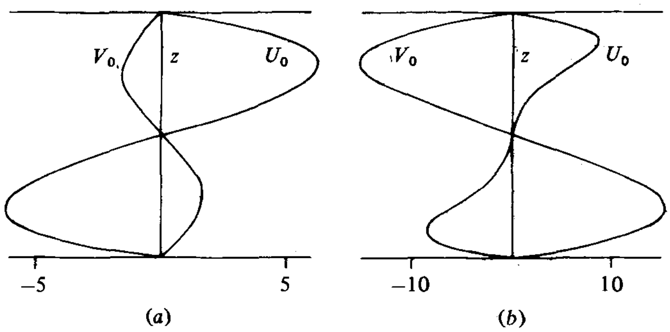

FIGURE 4. Typical profiles of the centrifugal functions U_0 and V_0 for rigid–rigid boundaries: (a) $\sigma = 1$, $T = 10^2$; (b) $\sigma = 1$, $T = 10^4$.

At order L^{-2} the solutions for w_1, v_1 and θ_1 contain components

$$(w_1, v_1, \theta_1) = (\alpha\bar{\Phi}(z), i\bar{V}(z), -\bar{\Theta}(z))\,e^{i\alpha r} + \dots \tag{5.6}$$

and the functions $\bar{\Theta}$, $\bar{V}$ and $\bar{\Phi}$ satisfy equations of the form (5.3) but with the three zero right-hand sides replaced by $\bar{F}_1$, $\bar{F}_2$ and $\bar{F}_3$, respectively, where

$$\begin{aligned}
\bar{F}_1 &= \tfrac{1}{4}\sigma^2 T G_0 s\alpha A_0 U_0 \Theta - \alpha\left(2\frac{\partial A_0}{\partial s} + \frac{A_0}{s}\right)\Theta,\\
\bar{F}_2 &= \tfrac{1}{4}\sigma T G_0 s\alpha A_0 (U_0 V - V_0'\Phi) - \alpha\left(2\frac{\partial A_0}{\partial s} + \frac{A_0}{s}\right)V + \frac{T^{\frac{1}{2}}}{\alpha}\left(\frac{\partial A_0}{\partial s} + \frac{A_0}{s}\right)\Phi',\\
\bar{F}_3 &= \tfrac{1}{4}\sigma T G_0 s\alpha A_0\left(U_0\Phi'' - U_0''\Phi - U_0\Phi\alpha^2 + \frac{1}{\alpha}\Theta'\right) - \alpha\left(2\frac{\partial A_0}{\partial s} + \frac{A_0}{s}\right)(\Phi'' - \alpha^2\Phi)\\
&\quad + \alpha\frac{A_0}{s}\Phi'' - \frac{1}{\alpha}\left(\frac{\partial A_0}{\partial s} + \frac{A_0}{s}\right)\Phi^{\text{iv}} + \frac{\partial A_0}{\partial s}(\alpha^3\Phi + \Theta).
\end{aligned} \tag{5.7}$$

Multiplication of the equation for Θ by $\bar{\Theta}$ and that for $\bar{\Theta}$ by Θ, subtraction and integration from $z = 0$ to $z = 1$ now yields

$$-\alpha R_c \int_0^1 (\bar{\Phi}\Theta - \Phi\bar{\Theta})\,dz = \int_0^1 \bar{F}_1 \Theta\,dz, \tag{5.8}$$

and from a similar treatment of the other equations

$$-T^{\frac{1}{2}} \int_0^1 (V\bar{\Phi}' - \Phi'\bar{V})\,dz = \int_0^1 \bar{F}_2 V\,dz, \tag{5.9}$$

and

$$T^{\frac{1}{2}} \int_0^1 (\bar{V}'\Phi - V'\bar{\Phi})\,dz + \alpha\int_0^1 (\bar{\Theta}\Phi - \Theta\bar{\Phi})\,dz = \int_0^1 \bar{F}_3 \Phi\,dz, \tag{5.10}$$

having invoked the boundary conditions (5.1). Combining these three results we find that in order that the solution at order L^{-2} be consistent we must have

$$\int_0^1 (R_c^{-1}\bar{F}_1\Theta - \bar{F}_2 V - \bar{F}_3\Phi)\,dz = 0. \tag{5.11}$$

Substitution from (5.7) shows that this reduces to

$$(\partial A_0/\partial s + A_0/2s)\,I = 0 \tag{5.12}$$

and so, by (5.5), is satisfied.

The crucial factor in the derivation of (5.12) from (5.11) is that only the even parts of $\bar{F}_1$ and $\bar{F}_3$, and the odd part of $\bar{F}_2$, are significant and therefore, since, as in the stress-free case of § 3, both U_0 and V_0 are odd, the centrifugal terms associated with G_0 in (5.7) make no contribution. Thus we may again proceed to order L^{-3} and envisage an amplitude equation for A of the same form as (4.1) although the coefficients of the various terms now have to be determined numerically from appropriate integrals of the various eigenfunctions Φ, Θ and V and the centrifugal terms U_0 and V_0. The determination of the boundary conditions for A also presents a considerable numerical task, for the eigensolutions of (5.3) are no longer orthogonal as in the stress-free case and so the simple set of equations obtained in (3.32) is now effectively replaced by an infinite set. However, in principle we may suppose that a value for A at $s = 1$ proportional to G_0 is again obtained.

If the lower surface of the container is rigid but the upper surface is stress-free it seems likely that the modifications to the analysis of §§ 3, 4 will be more severe since the system for the centrifugal velocity V_0 must now be solved subject to the boundary conditions

$$V_0' = V_0''' = 0 \quad (z = 1), \qquad V_0 = V_0' = V_0'' = 0 \quad (z = 0). \tag{5.13}$$

Here the condition $V_0'(0) = 0$ is again derived from the flux requirement $\int_0^1 U_0 dz = 0$. Thus U_0 and V_0 are no longer odd and so will almost certainly make a non-zero contribution in the condition (5.11) which will therefore no longer be consistent with (5.5). If this is the case it suggests that the appropriate magnitude for G_0 in (2.9) for the rigid–free case is $O(L^{-3})$ and the amplitude equation will contain a term $G_0\, isA$ in place of $G_0^2 s^2 A$. Moreover the boundary condition for A at $s = 1$ will be $A = 0$ to leading order and the actual centrifugal circulation will be relatively unimportant in this situation. It is hoped to consider these questions further in a future paper.

6. Discussion

We have considered the effect of centrifugal acceleration on the patterns of cellular convection which occur at the onset of thermal instability in a rotating container. The theory assumes axisymmetric convection of the type observed by Koschmieder (1967) and by appropriate choice of the order of magnitude of the rotational Froude number (1.1), restricts attention to the critical regime in which the motions due to the centrifugal force and the vertical instability are of equal magnitude. For free–free or rigid–rigid horizontal surfaces, the centrifugal acceleration is shown to have a double effect. The most significant is the smooth transition to finite amplitude convection represented by the non-zero boundary condition for the cellular amplitude function A, an effect prophesied by Koschmieder (1967) and clearly evident in the experimental results. The second represents the coupling of the vertical instability with centrifugal effects through the nonlinear terms in the equations of motion and is represented by the appearance of the new term proportional to $\gamma s^2 A$ in the amplitude equation, where γ

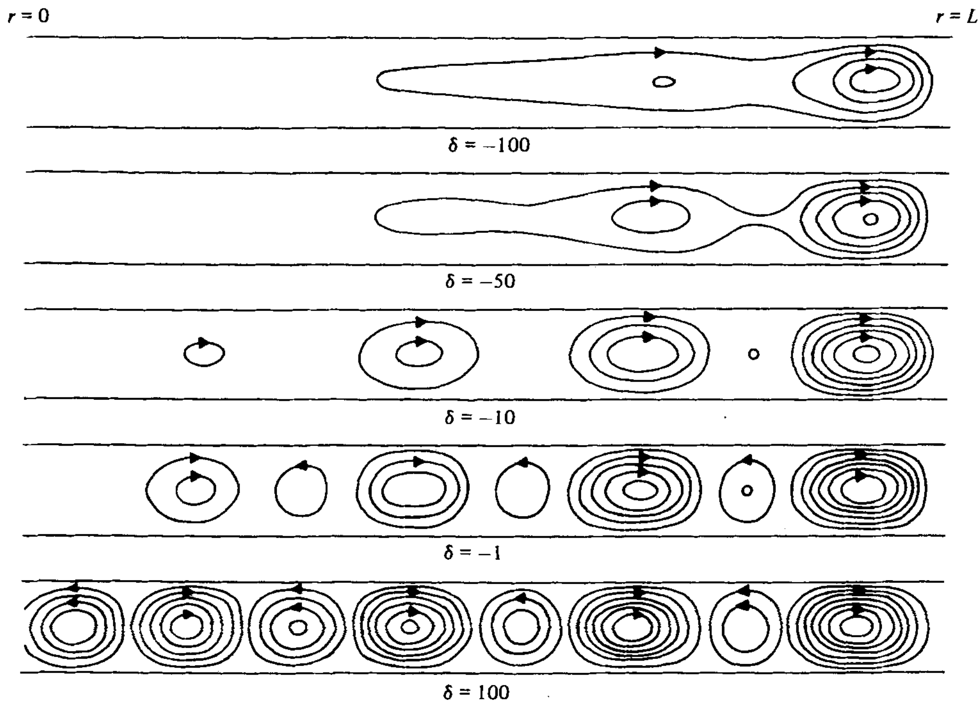

FIGURE 5. A typical transition to finite amplitude convection showing the streamlines in the cross-section of the container at intervals of 0·5, as determined by (3.6). The parameter values are $\sigma = 1$, $T = 10^2$, $G_0 = 0{\cdot}1$ and approximate solutions for A of the form

$$A = \lambda \sinh(-\delta)^{\frac{1}{2}} s/\sinh(-\delta)^{\frac{1}{2}} \quad \text{for} \quad \delta < 0,$$

and $A = \delta^{\frac{1}{2}} s^{\frac{1}{2}}$ for $\delta > 0$ are utilized.

may be positive or negative, depending on the values of the various parameters of the problem. This effect is more subtle and, as shown in § 4, in some parameter regimes can significantly affect the distribution of convection in the radial cross-section of the container, either enhancing or decreasing the effective local Rayleigh number, depending on the sign of γ. In fact in the stress-free case reference to figure 2 shows that, except for a small region of parameter space near $\sigma = 1$, γ is positive provided that T is greater than a critical value which is a function of σ, and the onset of convection is then advanced. The effect is greatest at large values of T where we may generally expect a significant drop in the value of the Rayleigh number at which convection is first observed, possibly providing a partial explanation of the highly subcritical motions observed in water at $T \sim 10^8$ by Rossby (1969).

Of the various experimental studies of convection in rotating systems only that of Koschmieder (1967) appears to provide details of the flow pattern at the onset of thermal instability. Other studies (e.g. Rossby 1969; Hudson, Abell & Tang 1978) tend to concentrate more on the heat flux properties of the system over wide parameter ranges. The flow pattern in the radial cross-section of the cylinder predicted by the present theory is a superposition of the centrifugal circulation and the motion due to the vertical instability, given to leading order by (3.6). A typical sequence of flow fields computed from (3.6) as the Rayleigh number increases is shown in figure 5. Only the interior solution has been used so that the solution ceases to be valid close to the vertical walls. However, the main features are remarkably similar to the series of flow patterns obtained by Koschmieder (1967), with a transition from a weak large-scale single-cell centrifugal circulation to a strong regularly spaced multi-cellular pattern. One of the unexplained features of the experiments was the appearance of the first counter-roll (i.e a roll in the opposite sense to the centrifugal circulation) at the outer wall of the cylinder, despite the fact that the centrifugal circulation is weakest at the centre. However this behaviour is indeed a feature of figure 5; the boundary condition on A at $s = 1$ has the effect of making the thermal convection strongest at the outer wall and it is for this reason that the first counter-roll can develop there.

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